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Final Exam, MTH 512, Fall 2019

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Score = $\frac{\quad}{100}$

$\frac{93}{100}$

QUESTION 1. (4 points) Let $T : V \to V$ be a linear transformation that is invertible, where V is an inner product vector space over R. Assume that $T^* = T^{-1}$. Assume that $T(v), T(w)$ are nonzero orthogonal elements of V for some nonzero elements $v, w \in V$. Convince me that v, w are orthogonal elements in V.

let $v, w \in V$ be non zero elements such that $T(v)$ and $T(w)$ are orthogonal

then $\langle T(v), T(w) \rangle = 0 \Rightarrow \langle v, T^*(T(w)) \rangle = 0$

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$\Rightarrow \langle v, T^{-1}(T(w)) \rangle = 0 \Rightarrow \langle v, w \rangle = 0 \Rightarrow$ v and w are orthogonal elements in V

QUESTION 2. (5 points) Let $T : V \to V$ be a linear transformation where V is a vector spaces over R and $IN(V) = 3$ (i.e., $dim(V) = 3$). Given $M = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 5 \\ 2 & 0 & 0 \end{bmatrix}$ is the matrix presentation of T with respect to an ordered basis $\{v_1, v_2, v_3\}$. Convince me that T is invertible. $\boxed{\text{Find } T^{-1}(v_3)}$. Convince me that $T^2 - 4T + 3I : V \to V$ is not invertible (singular).

• $|M| = +2(0-6) = -12 \neq 0 \Rightarrow M$ is invertible $\Rightarrow T$ is invertible

• $C_T(x) = x[x(x-3)] - 2[+2(x-3)] = x^2(x-3) - 4(x-3) = (x-3)(x^2-4)$
$= (x-3)(x-2)(x+2)$

thus 3 is an eigenvalue of T and
$\exists v \in V$ s.t $T(v) = 3v$.

Now let $F: V \to V$ s.t $F = T^2 - 4T + 3I$

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QUESTION 3. (4 points) Let $T : V \to V$ be a linear transformation. Consider the linear transformation $F = 2T^3 + 4T^2 + 512I : V \to V$. Let $W = Z(F)(Ker(F))$. Convince me that $T(w) \in W$ for every $w \in W$.

• let $w \in W \Rightarrow F(w) = 0$ for every $w \in W$

$\Rightarrow T(2T^3(w) + 4T^2(w) + 512w) = T(0)$

$\Rightarrow 2T^3(T(w)) + 4T^2(T(w)) + 512\,T(w) = T(0)$

$\Rightarrow F(T(w)) = 0$ (since T is linear transformation $\Rightarrow T(0) = 0$)

$\Rightarrow T(w) \in Z(F)$

$\Rightarrow T(w) \in W$ for every $w \in W$

QUESTION 4. Let $T : P_5 \to R^4$ such that $M_{B,B'} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 \\ 2 & 2 & 2 & -2 & -2 \\ 3 & 3 & 3 & -3 & -3 \end{bmatrix}$ be the matrix presentation of T with respect to $B = \{x^4, 1 + x^4, 1 + x + x^4, x^2 + x^4, x^3 + x^4\}$ and $B' = \{(1,1,1,1), (-1,1,0,1), (-2,-2,1,1), (-1,-1,-1,0)\}$.

(i) **(4 points)** Find the fake standard matrix presentation of T. (note that the Fake Matrix Presentation of T is with respect to $\{1, x, x^2, x^3, x^4\}$ and $\{e_1, e_2, e_3, e_4\}$). (you may use the available software)

$$\cancel{M} = \text{det } B_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 1 & -1 & -2 & -1 \\ 1 & 1 & -2 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\text{then } M = \underset{(4\times4)}{B_2} \underset{(4\times5)}{M_{B,B'}} \underset{(5\times5)}{B_1^{-1}} = \begin{pmatrix} 0 & 0 & 10 & 10 & -5 \\ 0 & 0 & 14 & 14 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 & 2 \end{pmatrix}$$

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(ii) **(3 points)** Write Range(T) as span of independent points.(you may use the available software)

$$\text{Range}(T) = \text{Col}(M) = \text{span}\{(10, 14, 0, -4)\}$$

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(iii) **(3 points)** Write $Z(T)(Ker(T))$ as span of some independent polynomials.(you may use the help of the available software)

$$Z(T): \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 0 & 0 & 10 & 10 & -5 & 0 \\ 0 & 0 & 14 & 14 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 & 2 & 0 \end{array} \Rightarrow \begin{cases} 10x_3 + 10x_4 - 5x_5 = 0 \\ 14x_3 + 14x_4 - 7x_5 = 0 \\ -4x_3 - 4x_4 + 2x_5 = 0 \end{cases}$$

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$\Rightarrow x_1, x_2, \cancel{x_3}, x_4$ free and $\Rightarrow x_5 = 2x_3 + 2x_4$

$\Rightarrow Z(T) = \{(x_1, x_2, x_3, x_4, 2x_3 + 2x_4) \mid x_1, x_2, x_3, x_4 \in R\} \Rightarrow$

(iv) **(2 points)** Find $T(\underbrace{5 + 2x - 4x^3}_{= w})$. Then find $T^{-1}(\underbrace{\cancel{5 + 2x - 4x^3}}_{w})$.

$$T(5 + 2x - 4x^3) = M \begin{pmatrix} 5 \\ 2 \\ 0 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} -40 \\ -56 \\ 0 \\ 16 \end{pmatrix}$$

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and $T^{-1}(\overset{w}{\cancel{5+2x-4x^3}}) = \{w + d \mid \cancel{T(w) = (5+2x}\ T(5+2x-4x^3) = w$ and $d \in Z(T)\}$

=

QUESTION 5. Let $T : R^3 \to R^3$ such that $T(1,0,1) = (1,1,1)$, $T(-1,1,1) = (-2,-2,-2)$, and $T(-1,0,1) \in Z(T)$. Consider the DOT PRODUCT on R^n.

(i) **(4 points)** Find $T^* : R^3 \to R^3$.

$\cdot\, T(1,0,0) = \frac{1}{2}[T(1,0,1) - T(-1,0,1)] = \frac{1}{2}(1,1,1) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$\cdot\, T(0,1,0) = T(-1,1,1) - T(-1,0,1) = (-2,-2,-2)$

$Q = (a,b,c) \in \mathbb{R}^3$ $\quad \cdot\, T(0,0,1) = \frac{1}{2}[T(-1,0,1) + T(1,0,1)] = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$\Rightarrow T^* = M^T Q \Rightarrow M^T Q^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -2 & -2 & -2 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \\ -2a - 2b - 2c \\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \end{pmatrix}$

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$\Rightarrow T^*(a,b,c) = (\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c,\ -2a-2b-2c,\ \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c)$

(ii) **(2 points)** write Range of T^* as span of some independent points.(you may use the help of the available software)

$T^*(Q) = M^T Q^T$: then Range$(T^*) = $ Col M^T

$= \text{span}\{(\frac{1}{2}, -2, \frac{1}{2})\}$.

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(iii) **(3 points)** Write $Z(T)$ as span of some independent points.(you may use the help of the available software)

$\cdot\, Z(T)$: $\left[\begin{array}{ccc|c} \frac{1}{2} & -2 & \frac{1}{2} & 0 \\ \frac{1}{2} & -2 & \frac{1}{2} & 0 \\ \frac{1}{2} & -2 & \frac{1}{2} & 0 \end{array}\right]$ $\Rightarrow \frac{1}{2}x_1 - 2x_2 + \frac{1}{2}x_3 = 0$

3 $\quad \Rightarrow \frac{1}{2}x_1 = 2x_2 - \frac{1}{2}x_3$

$\Rightarrow x_1 = 4x_2 - x_3$

$\Rightarrow Z(T) = \text{span}\{\overset{w_1}{(4,1,0)}, \overset{w_2}{(-1,0,1)}\}$

(iv) **(3 points)** Find $(Z(T))^\perp$ (i.e., find the subspace of R^3 that is orthogonal to $Z(T)$).(you may use the help of the available software) Stare at your answer in (ii) and your answer in (iv). Any connection.

$\cdot$ Let $m \in (Z(T))^\perp \overset{\text{def}}{\Rightarrow} \vec{m} = (a,b,c)$ 3

then $\langle m, w_1 \rangle = 0 \Rightarrow (a,b,c).(4,1,0) = 0 \Rightarrow 4a + b = 0$

$\langle m, w_2 \rangle = 0 \Rightarrow (a,b,c).(-1,0,1) = 0 \Rightarrow -a + c = 0$

$\Rightarrow a = c$ and $b = -4a$

$\Rightarrow (Z(T))^\perp = \{(a, -4a, a) \mid a \in \mathbb{R}\}$

$= \text{span}\{(1,-4,1)\}$. (its equal to Range (T^*))

QUESTION 6. (5 points) Consider the normal dot product on R^n. Let A be a symmetric matrix over R. Convince me that all eigenvalues of A are real.

Let $T: V \to V$ be a linear transformation such that $T(Q) = AQ^T$ for any non-zero $Q \in V$.

$\Rightarrow \langle T(Q), Q\rangle = \langle AQ^T, Q\rangle = (AQ^T)^T \cdot Q^T = Q A^T Q^T = \langle Q, A^T Q^T\rangle = \langle Q, T^*(Q)\rangle$

$\Rightarrow T^*(Q) = A^T Q^T = AQ^T = T(Q) \Rightarrow$ (T is also symmetric).

Let α be any eigenvalue of $A \Rightarrow \alpha$ is any eigenvalue of T (ie $T(v) = \alpha v$)

$\Rightarrow$ • $\langle T(v), v\rangle = \langle \alpha v, v\rangle = \alpha\langle v, v\rangle$

• $\langle T(v), v\rangle = \langle v, T^*(v)\rangle = \langle v, T(v)\rangle = \langle v, \alpha v\rangle = \bar{\alpha}\langle v, v\rangle$

$\Rightarrow \alpha = \bar{\alpha}$ since $\langle v, v\rangle \neq$
$\Rightarrow \alpha$ is real.

QUESTION 7. (5 points) Let $T : V \to V$ be a linear transformation. Assume that $T^2 = T$. Convince me that $Range(T) \cap Z(T) = 0_v$.

• Let $x \in Range(T) \cap Z(T)$

$\Rightarrow x \in Range(T)$ and $x \in Z(T)$

$\Rightarrow \exists\, y \in V$ s.t $T(y) = x$ and $T(x) = 0$

$\Rightarrow T(T(y)) = T(x) \to T^2(y) = T(x) \Rightarrow T(y) = T(x)$

$d = y - x \in Z(T)$
$\Rightarrow y = d + x \in Z(T)$ since $x \in Z(T)$

$\Rightarrow T(y) - T(x) = 0 \Rightarrow T(y - x) = 0 \Rightarrow y - x \in Z(T) \Rightarrow y \in Z(T)$

$\Rightarrow x = T(y) = 0_v \Rightarrow Range(T) \cap Z(T) = 0_v$.

QUESTION 8. (4 points) Consider the normal dot product on R^n. Let A be a matrix (of course $n \times n$) such that $A^T = A$ over R. Assume that for some nonzero points V and W in R^n, we have $AV^T = aV^T$ and $AW^T = bW^T$ for some real numbers a, b such that $a \neq b$. Convince me that V and W are orthogonal.

Define a linear transformation: $T: V \to V$ such that $T(Q) = AQ^T$ for some $Q \in V$

and as $A^T = A$ (symmetric) $\Rightarrow T = T^*$ (proved in question 6).

and assume that $\langle V, W\rangle \neq 0$

$\Rightarrow$ • $\langle T(V), W\rangle = \langle AV^T, W\rangle = \langle aV^T, W\rangle = a\langle V, W\rangle$

• $\langle T(V), W\rangle = \langle V, T^*(W)\rangle = \langle V, T(W)\rangle = \langle V, bW^T\rangle$

$= \bar{b}\langle V, W\rangle = b\langle V, W\rangle$ (real)

$\Rightarrow a\langle V, W\rangle = b\langle V, W\rangle$ where $\langle V, W\rangle \neq 0$

$\Rightarrow a = b$ (contradiction) $\Rightarrow \langle V, W\rangle = 0 \Rightarrow V$ and W are orthogonal

QUESTION 9. (5 points) Consider the normal dot product on R^n. Let A be a matrix (of course $n \times n$) such that A is nonsingular (i.e., invertible) and $A^T = A$ over R. Let $B = A^2$. Convince me that $B^T = B$, B is invertible, and all eigenvalues of B are real and each eigenvalue is strictly larger than 0 (i.e., B is positive definite)

Let. $B = A^2 \Rightarrow B^T = (A.A)^T = A^T.A^T = A.A = A^2 = B^T$

define $T: V \to V$ a linear transformation s.t $T(Q) = A Q^T$

and $F: V \to V$ a linear transformation s.t $F(Q) = T^2(Q) = A^2 Q^T$
$= B Q^T$.

. As $B^T = B \Rightarrow$ all eigenvalues of B are real (proved in question 6)

let α be any eigenvalues of $B \Rightarrow \alpha$ is an eigenvalue of $F \Rightarrow F(v) = \alpha v$.

$\alpha \langle v, v \rangle = \langle \alpha v, v \rangle = \langle F(v), v \rangle = \langle T^2(v), v \rangle = \langle T(T(v)), v \rangle = \langle T(v), T^*(v) \rangle$
$= \langle T(v), T(v) \rangle > 0$

$\alpha \langle v, v \rangle > 0$

$\Rightarrow \alpha > 0$ since $\langle v, v \rangle >$

$\Rightarrow B$ is invertible since $\alpha \neq 0$

QUESTION 10. Let $J = J_{-1}^{(2)} \oplus J_2^{(2)} \oplus J_{-1} \oplus J_2$ be the Jordan form of a matrix A.

(i) **(3 points)** Find $C_A(x)$

$C_A(x) = (x+1)^3 (x-2)^3$ ✓

(ii) **(3 points)** Find $m_A(x)$

$m_A(x) = (x+1)^2 (x-2)^2$ ✓

(iii) **(3 points)** For each eigenvalue a of A find $IN(E_a)$ (i.e., $dim(E_a)$).

$IN(E_{-1}) = 2$ and $IN(E_2) = 2$ ✓

(iv) **(3 points)** Find the rational form of A.

$R_A = C(f_1) \oplus C(f_2) \oplus C(f_3) \oplus C(f_4)$ where $f_1 = (x+1)^2 = x^2 + 2x + 1$
$f_2 = (x-2)^2 = x^2 - 4x + 4$
$f_3 = x+1$
$f_4 = x - 2$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$ ✓

(v) **(3 points)** Is A diagnolizable? explain?

No, A is not diagonalizable since $m_A(x) \neq (x+1)(x-2)$
(i.e $m_A(x)$ must be done without the repetition).

QUESTION 11. Let $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(i) **(3 points)** Find $C_A(x)$ (you may use the available software calculators) OR find it by HAND.

$C_A(x) = |xI - A| = (x-3)(x+3)(x-1)^2$

(ii) **(4 points)** Find $m_A(x)$ (you may use the available software calculators) OR find it by hand (maybe LONG)

Now $m_A(x)$ could be $(x-3)(x+3)(x-1)^2$ or $(x-3)(x+3)(x-1)$ ✗

but here $m_A(x) = (x-3)(x+3)(x-1)^2$ (by substituting A in both)

(iii) **(3 points)** Find the Rational Form of A

$R_A = C[(x-3)] \oplus C[(x+3)] \oplus C[(x^2-2x+1)]$

$= \begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ 0 & \boxed{-3} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ 0/3

(iv) **(3 points)** Find the Jordan Form of A

$J = J_3 \oplus J_{-3} \oplus J_1^{(2)} = \begin{bmatrix} \boxed{3} & 0 & 0 & 0 \\ 0 & \boxed{-3} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ✓

QUESTION 12. (5 points) Let $T : V \to V$ be a linear transformation that is invertible, where V is a finite dimensional inner product vector space over R. Assume that $T^* = -T$. Convince me that

$$C_T(x) = (x^2 + a_1)^{n_1}(x^2 + a_2)^{n_2} \cdots (x^2 + a_m)^{n_m}$$

, where $a_1, a_2, ..., a_m$ are distinct nonzero positive real numbers, and $n_1, ..., n_m$ are positive integers.

• let $v \in V$ and α be the eigenvalue of T 4/5

then, $\langle T(v), v \rangle = \langle \alpha v, v \rangle = \alpha \langle v, v \rangle$

, $\langle T(v), v \rangle = \langle v, T^*(v) \rangle = \langle v, -T(v) \rangle = \langle v, -\alpha v \rangle = -\bar{\alpha} \langle v, v \rangle$

$\Rightarrow \alpha = -\bar{\alpha} \Rightarrow$ ~~...~~ α is ~~pure~~ pure imaginary.

so all eigenvalues of T must be pure imaginary.

why not 0 ⟶

QUESTION 13. (5 points) Let $T : R^3 \to R^3$ be a nonzero non-diagnolizable linear transformation. Given $T^3 - 4T^2 + 4T = 0$. Find all Jordan forms of the standard matrix presentation of T. Find all Rational forms of the standard matrix presentation of T.

$$T^3 - 4T^2 + 4T = 0 \Rightarrow T(T^2 - 4T + 4) = 0 \Rightarrow T(T - 2I)^2 = 0$$

(we Know that $C_T^{(x)}|_T = 0$)

then $C_T(x) = x(x-2)^2$ and $m_T(x) = x(x-2)^2$ (since T is non-diagonaliz so $m_T(x) \neq x(x-2)$)

$$\Rightarrow J_T = J_0 \oplus J_2^{(2)} \quad \text{and} \quad R_T = C[0] \oplus C[x^2 - 4x + 4]$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 1 & 4 \end{pmatrix}$$

QUESTION 14. (6 points) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. Find the SMITH form of A over Z (i.e., find invertible matrices R, C over Z and a diagonal matrix D over Z (with special property as explained in class) such that $D = RAC$)

before that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$C_1 \leftrightarrow C_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$-4R_1 + R_2 \to R_2$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$-2C_1 + C_2 \to C_2$

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$-R_2$

$$\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = R \qquad \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = D \qquad \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = C$$

and $D = RAC$.

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